Problem 7

A peach pie is taken out of the oven at 5:00 PM. At that time it is piping hot, 100°C. At 5:10 PM its temperature is 80°C; at 5:20 PM it is 65°C. What is the temperature of the room?

Solution

Newton's law of cooling states that the rate the pie's temperature decreases in time is proportional to the difference between the pie's temperature and the temperature of the room. Mathematically this is written as

$$\frac{dT}{dt} \propto -(T - T_{\text{room}}).$$

To change this proportionality to an equation, we must introduce a constant of proportionality, k.

$$\frac{dT}{dt} = -k(T - T_{\text{room}})$$

Solve for T by separating variables.

$$\begin{aligned}
\frac{dT}{T - T_{\text{room}}} &= -k\,dt \\
\ln|T - T_{\text{room}}| &= -kt + C \\
|T - T_{\text{room}}| &= e^{-kt+C} \\
T - T_{\text{room}} &= Ae^{-kt} \\
T(t) &= Ae^{-kt} + T_{\text{room}}
\end{aligned}$$

Use the given conditions in the problem, $T(0) = 100$, $T(10) = 80$, and $T(20) = 65$, to determine the constants.

$$\begin{aligned}
T(0) &= A + T_{\text{room}} = 100 \quad \rightarrow \quad T_{\text{room}} = 100 - A \\
T(10) &= Ae^{-10k} + T_{\text{room}} = 80 \\
T(20) &= Ae^{-20k} + T_{\text{room}} = 65
\end{aligned}$$

Substituting the first equation into the second and third gives

$$\begin{aligned}
A(1 - e^{-10k}) &= 20 \\
A(1 - e^{-20k}) &= 35
\end{aligned}$$

These two equations imply that

$$\frac{20}{1 - e^{-10k}} = \frac{35}{1 - e^{-20k}}.$$

Solving for k gives

$$k = \frac{1}{10} \ln \frac{4}{3}.$$

And so

$$A = 80 \quad \text{and} \quad T_{\text{room}} = 20°\text{C}$$

The temperature of the pie t minutes after 5:00 PM is thus

$$T(t) = 20\left(4e^{-\frac{t}{10}\ln\frac{4}{3}} + 1\right)$$

or

$$T(t) = 20\left[4\left(\frac{3}{4}\right)^{t/10} + 1\right]$$

The graph of this function is shown below.

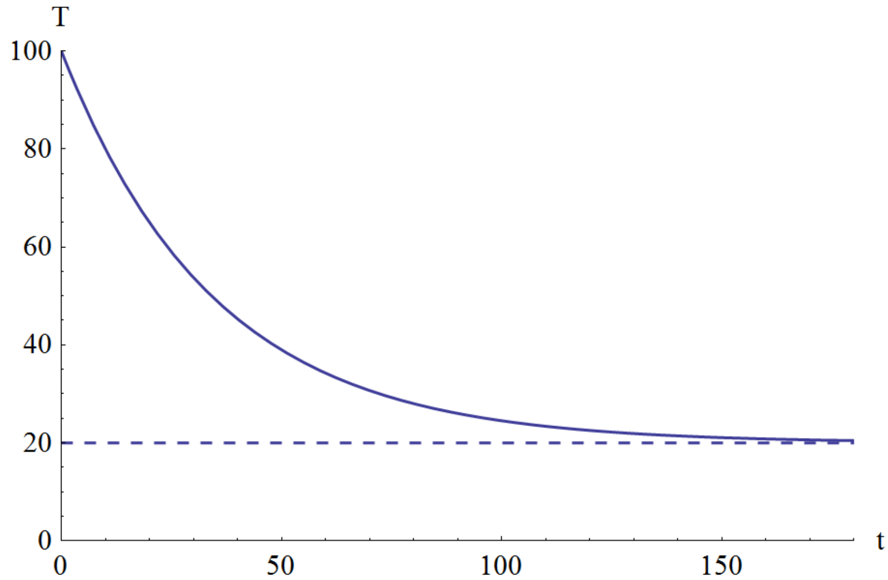

Figure 1: Plot of $T(t)$ vs. t over the course of 3 hours.

The temperature of the room is 20°C.